International Journal of Theoretical Physics, Vol. 34, No. 8, 1995

Representations of D-Posets

Sylvia Pulmannová[1] and Alexander Wilce[2]

Received November 21, 1994

A generalization of an orthoalgebra, which includes the set of all effects (i.e., s.a. operators between 0 and 1 on a Hilbert space) is a D-poset or an effect algebra, equivalently. Two generalizations of test spaces the logics of which are D-posets are investigated and their equivalence is shown.

INTRODUCTION

D-posets,[3] introduced in Kôpka and Chovanec (1994), are generalizations of orthoalgebras in which one drops the requirement that no nonzero element be self-orthogonal. Orthoalgebras arise as the logics of algebraic test spaces (i.e., manuals), and it is natural to wonder whether D-posets have an analogous representation. Two such representations have recently been described by the authors (Dvurečenskij and Pulmannová, 1994; Wilce, 1994). The purpose of this note is compare these two constructions and show that they are essentially isomorphic.

1. ALGEBRAIC SETS IN A PARTIAL ABELIAN SEMIGROUP

By a *partial Abelian semigroup* (PAS) we mean a set S together with a partially defined commutative and associative binary operation $\oplus$.[4] Define $a \perp b$ iff $a \oplus b$ is defined. For convenience, we assume that S contains a *zero*, i.e., a distinguished element 0 such that $0 \oplus a = a$ for all $a \in S$. We

[1] Mathematical Institute, Slovak Academy of Sciences, Bratislava, Slovakia.
[2] Department of Mathematics, University of Pittsburgh/Johnstown, Johnstown, Pennsylvania.
[3] Called *effect algebras* in Foulis (1994) and Foulis and Bennett (1994) and *D-algebras* in Wilce (1994).
[4] That is, if $a \oplus b$, resp. $a \oplus (b \oplus c)$, is defined, then so is $b \oplus a$, resp. $(a \oplus b) \oplus c$, and the two are equal.

0020-7748/95/0800-1689$07.50/0

say that S is *positive* iff $a \oplus b = 0$ implies $a = b = 0$, and *cancelative* iff $a \oplus b = a \oplus c$ implies $b = c$ for all $a, b, c \in S$.

Any positive, cancelative PAS is partially ordered by the relation $a \leq b$ iff $a \oplus x = b$ for some $x \in L$. A *D-poset* is a positive, cancelative PAS L having a largest element 1—in other words, an element such that $\forall a \in L$, $\exists!\, b \in S$ with $a \oplus b = 1$ (note that such an element, if it exists, is unique). A D-poset in which $a \perp a \Rightarrow a = 0$ is an *orthoalgebra*.

In Hedlíkova and Pulmannová (1994) the notion of a generalized difference poset has been introduced as follows.

Let $(P, \leq)$ be a poset with the smallest element 0 and let $\ominus$ be a partial binary operation on P such that $b \ominus a$ is defined iff $a \leq b$. Then $\ominus$ is a difference on $(P, \leq)$ if and only if the following two conditions are satisfied for all $a, b, c \in P$:

(1) $a \ominus 0 = a$.
(2) If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

A difference $\ominus$ is called *cancelative* if the following condition is satisfied:

(C) If $a \leq b, c$ and $b \ominus a = c \ominus a$, then $b = c$.

A poset with a cancelative difference containing a smallest element 0 is a *generalized difference poset* (GDP).

Let $(P, \leq, \ominus)$ be a poset with a difference satisfying condition (C). This means that for every $a, b \in P$ there is at most one $c \in P$ such that $a = c \ominus b$. Thus property (C) enables us to define a sum operation on P, that is, a partial binary operation $\oplus$ on P given by $(a, b, c \in P)$:

(S) $a \oplus b$ is defined and $a \oplus b = c$ if and only if $c \ominus b$ is defined and $a = c \ominus b$.

By Hedlíkova and Pulmannová (1994), Corollary 1.13, a cancelative positive PAS with a zero coincides with a generalized difference poset; and a unital cancelative positive PAS with a zero coincides with a difference poset in the sense of Kôpka and Chovanec (1994).

A *congruence*[5] on a PAS is an equivalence relation $\sim$ such that for all $a, b, c \in S$, if $a \sim b$ and $a \oplus c$ is defined, then so is $b \oplus c$, and $a \oplus c \sim b \oplus c$. Let S be any PAS. Call a subset M of S *dominating* iff $\forall a \in S$, $\exists b \in S$ with $a \oplus b \in M$, and *algebraic* iff the relation

$$a \sim_M b \Leftrightarrow \exists x \in S\; a \oplus x,\, x \oplus b \in M$$

is a congruence. If M is algebraic, we write S/M for $S/\sim_M$, and $[a]_M$ for the

[5]Called a faithful congruence in Wilce (1994).

congruence class of $a \in S$ in S/M. It can be shown that $S/\sim$ is always a cancelative, unital PAS under the inherited operation $[a]_M \oplus [b]_M = [a \oplus b]_M$. In particular, an algebraic set is dominating (Wilce, 1994, Lemma 3.2).

The following is useful in establishing that a set is algebraic:

Lemma 1. Let M be dominating in S. If, for all $a, b, c \in S$, $a \sim_M b$ and $b \oplus c \in M$ imply $a \oplus c \in M$, then M is algebraic.

Proof. See Wilce (1994), Lemma 3.4. ■

By way of example, if S is the collection of subsets of a set X dominated by a fixed covering $\mathcal{A}$ of X, then $M = \mathcal{A}$ is algebraic iff $(X_1\ \mathcal{A})$ is an algebraic test space (Dvurečenskij and Pulmannová, 1994; Foulis, 1994), and in this case, $S/M \simeq \Pi(\mathcal{A})$, the logic of $(X, \mathcal{A})$. Such a logic is an orthoalgebra, and conversely, every orthoalgebra arises canonically in this way (Gudder, 1988). In the next section, we consider two (essentially isomorphic) constructions leading to a canonical representation for arbitrary D-posets.

If S is a PAS and $M \subseteq S$, we say that M is *irredundant* iff $\forall a, b \in M$, $b = a \oplus x \Rightarrow x = 0$.

Lemma 2. Let S be a positive PAS and $M \subseteq S$ an irredundant set. Then M is algebraic iff M is dominating and for all $a, b, c \in S$,

$$a \sim_M b \perp c \Rightarrow a \perp c$$

Proof. If M is algebraic, it is dominating and, as $\sim_M$ is a congruence, the condition above holds. Conversely, suppose M is irredundant and dominating and that $a \sim_M b \perp c \Rightarrow a \perp c$ for all $a, b, c \in M$. We shall show that M satisfies the hypothesis of Lemma 1. Suppose that $a \sim_M b$ and $b \oplus c \in M$. Then $a \oplus c$ exists; hence, as M is dominating, there exists x with $a \oplus c \oplus x \in M$. As $a \sim_M b$, there is some d such that $a \oplus d$, $d \oplus b \in M$; then $d \sim c$, and so $d \perp (a \oplus x)$. Again, since M is dominating, there exists some y such that $d \oplus (a \oplus x) \oplus y = a \oplus d \oplus (x \oplus y) \in M$. But $a \oplus d \in M$, and M is irredundant, so $x \oplus y = 0$. Since S is positive, $x = 0$, whence $a \oplus c \in M$, as desired. ■

If S is positive and M is an irredundant algebraic set, then for all $a \in M$ and all $x \in S$, $x \perp a \Rightarrow x = 0$. Hence, $[a]_M \oplus [b]_M = 0$ iff $[a]_M, [b]_M = 0$. Thus, S/M is a D-poset.

What distinguishes a general D-poset from an orthoalgebra is the possibility that a nonzero element $a \in L$ may have multiplicity greater than 1. Indeed, if S is any PAS, we may set

$$\mu(a) = \sup\{n \in \mathbb{N} \mid n \cdot a = a \oplus \cdots \oplus a \ (n \text{ times}) \text{ exists}\}$$

Call $\mu(a)$ the *multiplicity* of a (noting that it may equal ω). We say that a function f: $S \to \mathbb{N}$ is *summable* iff $\oplus_{a\in S} f(a)\cdot a$ exists. This requires f to be finitely nonzero and $f(a) \leq \mu(a)$ for all a. Let $F(S)$ be the collection of summable functions on S, and note that this is a PAS under pointwise addition (where this is defined). The following is a slight modification of Theorem 4.2 and Lemma 3.5 in Wilce (1994).

Lemma 3. If N is an algebraic subset of S and M is the set of summable functions $f \in F$ with $\oplus_a f(a)\cdot a \in N$, then M is algebraic in F and $F/M \simeq S/N$.

As a particular example, let L be a D-poset, and take $N = \{1\}$. Then $L \simeq L/N$, and M consists of the collection of summable functions with sum 1. By Lemma 3, then, $F(L)/M \simeq L$. Thus, all D-posets arise canonically from an algebraic set of integer-valued functions.

If L is an orthoalgebra, then $\mu(a) = 1$ for all $a \neq 0$. Thus, $F(L)$ consists of $\{0, 1\}$-valued functions, i.e., of summable subsets of $L\backslash\{0\}$. In this case, then, $F(L)$ is exactly the manual of orthopartitions of L, and the isomorphism $F(L)/M \simeq L$ is the canonical one.

Let us call an algebraic subset M of a PAS S *D-algebraic* iff L/M is a D-poset. Note that this is equivalent to L/M being positive.

We will call a subset M of a PAS S *an order-filter* iff $x \in M$, $y \perp x$ imply $x \oplus y \in M$.

Lemma 4. Let $M \subseteq S$ be algebraic. If M is an order-filter, then it is D-algebraic.

Proof. It suffices to show that if $a \oplus b \sim_M 0$, then $a \sim_M 0$. But $a \oplus b \sim_M 0$ iff there exists some $c \in M$ with $a \oplus b \oplus c \in M$; in this case, $a \oplus b \perp c$, whence $b \perp c$. Since M is an order-filter, $b \oplus c \in M$, and it follows that $a \sim_M 0$. ∎

Observe that $S/M = \{0\}$ iff $x \oplus y \in M$ for some $x, y \in M$. Since $[a] \perp [b]$ implies $a \perp b$ by the definition of a congruence, this may occur only if the operation $\oplus$ in S is totally defined.

We now show that the order-filter generated by any algebraic set is D-algebraic.

Theorem 1. Let S be a PAS and let M be an algebraic subset of S. Then

$$M^1 := \{a \in S: a \sim_M x \oplus y, x \in M\}$$

is D-algebraic.

Proof. In what follows, $\sim$ denotes the perspectivity with respect to M, and $\sim^1$ the perspectivity with respect to M^1. Since M^1 is an order-filter, it suffices by Lemma 4 to show that M^1 is algebraic. Since M is dominating, so is M^1, so by Lemma 1, it suffices to show that if $a \sim^1 b$ and $b \oplus c \in M^1$, then $a \oplus c \in M^1$. Thus, suppose that for some $z \in S$, $a \oplus z \in M^1$, $b \oplus z \in M$. Since M is dominating, there are z_1, z_2, z_3 such that $a \oplus z \oplus z_1 \in M$, $b \oplus z \oplus z_2 \in M$, $b \oplus c \oplus z_3 \in M$. It follows that $a \oplus z_1 \sim b \oplus z_2$, $c \oplus z_3 \sim z \oplus z_2$.

On the other hand, since $a \oplus z$, $b \oplus z \in M^1$, there are $x, y \in M$ and $u, v \in S$ such that $x \oplus u \sim a \oplus z$, $y \oplus v \sim b \oplus z$. It then follows that $a \oplus z \sim x \oplus u \sim a \oplus z \oplus z_1 \oplus u$, hence $z_1 \oplus u \sim 0$. Similarly, $b \oplus z \sim y \oplus v \sim b \oplus z \oplus z_2 \oplus v$ implies $z_2 \oplus v \sim 0$.

Now observe that if $x \oplus y \sim 0$, then for any $s \in S$, $s \perp x \oplus y$, hence $s \perp x$. In particular, z_1 and z_2 are orthogonal to every element of S, and we may have

$$a \oplus z \oplus z_1 \oplus z_2 \sim a \oplus z_1 \oplus c \oplus z_3 \sim b \oplus z_2 \oplus c \oplus z_3$$

This entails $a \oplus c \oplus z_1 \sim b \oplus c \oplus z_2$, which implies $a \oplus c \oplus z_1 \oplus u \sim b \oplus c \oplus z_2 \oplus u$, and since $z_1 \oplus u \sim 0$, we get $a \oplus c \sim b \oplus c \oplus z_2 \oplus u$. Now $b \oplus c \in M^1$ implies $b \oplus c \oplus z_2 \oplus u \in M^1$, which gives $a \oplus c \in M^1$, as desired. ■

2. PARTIAL FUNCTIONS AND D-TEST SPACES

A function $f: X \to \mathbb{Z}_+$ is sometimes interpreted as a "multiset," i.e., an object analogous to a set, but allowing an element to occur with (finite) multiplicity greater than 1. Thus, for instance, the collection of summable functions on a PAS may be understood as a collection of multisets.

More generally, we shall speak of a pair (X, F) consisting of a set X and a collection F of integer-valued functions $f: X \to \mathbb{Z}_+$ as a *generalized test space*. When no confusion can result, we refer simply to the *generalized test space* X, leaving F tacit.

We refer to a function g with $0 \leq g \leq f$ for some $f \in F$ as an *event* for F, and denote by $\mathscr{E}(X)$ the collection of all events. We note that $\mathscr{E}(X)$ is a positive, cancelative PAS under the operation $(f \oplus g)(x) = f(x) + g(x)$ provided that $f + g \in \mathscr{E}(X)$. We say that X is *algebraic* (*D-algebraic*) iff F is algebraic (D-algebraic) in $\mathscr{E}(X)$. In the D-algebraic case, $\mathscr{E}(X)/F$ is a D-poset, which we call the *logic* of X, denoting it by $\Pi(X)$. We note that an algebraic set F may be replaced by a D-algebraic set F^1, if necessary.

Another representation for a multiset involves replacing points by sets. Specifically, if f is a surjection onto a set E, we may regard $f^{-1}(x)$ as a set of "copies" of $x \in E$. From this point of view, a multiset of elements of a set X is a partial function $f: I \to X$, where I is some (suitably large) set I.

Let $\mathbb{P}(I, X)$ denote the collection of partial functions from I to X, that is, the collection of sets $f \subseteq I \times X$ such that $(i, x), (i, y) \in f \Rightarrow x = y$ for all $i \in I$. For $f \in \mathbb{P}(I, X)$, let $\text{dom}(f) = \{i \in I | \exists x \in X, (i, x) \in f\}$ and $\text{ran}(f) = \{x \in X | \exists i \in I, (i, x) \in f\}$. For $i \in \text{dom}(f)$, we write $f(i)$ for the unique $x \in X$ with $(i, x) \in f$. Note that if $\#(X) = 1$, then $\mathbb{P}(I, X)$ may be identified in a natural way with $\mathcal{P}(I)$ via $f \mapsto \text{dom}(f)$. $\mathbb{P}(I, X)$ becomes a PAS if we set $f \perp g$ iff $\text{dom}(f) \cap \text{dom}(g) = \emptyset$ and let $f \oplus g = f \cup g$ in this case.

We shall say that $f \in \mathbb{P}(I, X)$ has *finite multiplicity* iff $\#f^{-1}(x) < \infty$ for all $x \in X$. The set $\mathbb{F}(I, X)$ of partial functions from I to X with finite multiplicity is a sub-PAS of $\mathbb{P}$. Moreover, we have a natural map

$$\phi: \mathbb{F}(I, X) \to \mathbb{Z}_+^X$$

given by $\phi(f) = \#f^{-1}(\cdot)$. Thus, every partial function in $\mathbb{F}(I, X)$ gives rise to an integer-valued function on X. Note, too, that if $\text{dom}(f) \cap \text{dom}(g) = \emptyset$, then

$$\phi(f) + \phi(g) = \phi(f \oplus g)$$

Thus, ϕ is a homomorphism from the PAS $\mathbb{F}(I, X)$ into the semigroup $\mathbb{Z}_+^X$ (with pointwise addition).

One may preorder $\mathbb{P}(I, X)$ by the relation $F \preceq G$ iff there exists an injection σ: $\text{dom}(F) \to \text{dom } G$ such that $F = G \circ \sigma$. Equivalently, $F \preceq G$ iff $\#F^{-1}(x) \leq \#G^{-1}(x)$ for all $x \in X$. Functions F and G are *equivalent* iff $F \preceq G \preceq F$, in which case we write $F \approx G$. Evidently,

$$F \approx G \Leftrightarrow \phi(F) = \phi(G)$$

Therefore, if we work with equivalence classes of partial functions, we are in effect dealing with $\mathbb{Z}_+$-valued functions.

In Dvurečenskij and Pulmannová (1994) a *D-test space* is defined to be a collection $\mathcal{T}$ of $\approx$-equivalence classes of partial functions F: $I_F \to X$ such that (i) for all $x \in X$, $\exists [F] \in \mathcal{T}$ with $x \in \text{ran}(F)$ and (ii) $F^{-1}(x)$ is finite for all $x \in \text{ran}(F)$. $\mathcal{T}$ is *irredundant* iff for all $F, G \in \mathcal{T}$, $\#F^{-1}(x) \leq \#G^{-1}(x)$ for all $x \in X$ implies $F = G$.[6]

Let $(X, \mathcal{T})$ be a D-test space. There is no harm in setting $I = \bigcup_{[F] \in \mathcal{T}} I_F$ and treating the functions F as elements of $\mathbb{F}(I, X)$. An *event* for $(X, \mathcal{T})$ is the $\approx$-equivalence class $[G]$ of a partial function $G \preceq F$. The class of events of $(X, \mathcal{T})$ is denoted by $\mathcal{E}(\mathcal{T})$. Clearly, there is a bijective correspondence $[F] \mapsto \phi(F) = \#F^{-1}$ between $\mathcal{T}$ and a certain class of functions $X \to \mathbb{Z}_+$. If we let $E(X)$ denote the image of $\mathcal{E}(X, \mathcal{T})$ under ϕ, we obtain a PAS of $\mathbb{Z}_+$-valued functions containing $\phi(\mathcal{T})$. It is easily checked that the notions of

[6] In Dvurečenskij and Pulmannová (1994) it is assumed that all D-test spaces are irredundant, but it will prove convenient to drop this requirement. Thus, our definition is, strictly speaking, a bit more general than that given there.

orthogonality and perspectivity for events $[G]$ and $[H]$ as defined in Dvurečenskij and Pulmannová (1994) are equivalent to the notions of the same name for elements $\phi(G)$ and $\phi(H)$ of the PAS $E(X)$. Therefore, every D-test space $(X, \mathcal{T})$ may be reorganized into a generalized test space $(X, \phi(\mathcal{T}))$, and ϕ provides an isomorphism between the event structures preserving local complements and, hence, perspectivity.

Conversely, we have the following:

Lemma 5. Let (X, F) be any generalized test space. Then there exists a D-test space $(X, \mathcal{T})$ such that $F = \phi(\mathcal{T})$.

Proof. For each $f \in F$ and $x \in X$, let $I_{f,x}$ be a set with cardinality $f(x)$—in particular, $I_{f,x} = \emptyset$ if $f(x) = \emptyset$. Taking the sets $I_{f,x}$ to be pairwise disjoint as (x, f) ranges over $X \times F$, set $I_f = \cup_{x \in X} I_{f,x}$. Let F_f: $I_f \to X$ be given by $F_f(i) = f(x)$, where $i \in I_{f,x}$. Then $\mathcal{T} = \{F_f | f \in F\}$ is a D-test space and $\phi([F_f])(x) = \#F_f^{-1}(x) = f(x)$ by construction. ■

Note that, by Lemma 2, an irredundant D-test space $(X, \mathcal{T})$ is algebraic in the sense of Dvurečenskij and Pulmannová (1994) iff $\phi(\mathcal{T})$ is algebraic in the sense defined above.

In Dvurečenskij and Pulmannová (1994), morphisms between D-test spaces are defined as follows. If ϕ: $X \to Y$ and $F \in \mathbb{P}(I, X)$, then $\phi(F) := \phi \circ F \in \mathbb{P}(I, Y)$. If $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are D-test spaces, we call ϕ a morphism between X and Y iff $[\phi(F)] \in \mathcal{U}$ for all $[F] \in \mathcal{T}$. We have

$$\#\phi(F)^{-1}(y) = \#F^{-1}\phi^{-1}(y) = \sum_{\phi(x)=y} \#F^{-1}(x)$$

Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be generalized test spaces. We say that ϕ: $X \to Y$ is a morphism iff, for every $f \in \mathcal{F}$, the function $\phi(f)$ defined by

$$\phi(f)(y) = \sum_{\phi(x)=y} f(x)$$

belongs to $\mathcal{G}$.

Note that if $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ are test spaces, then for each $E \in \mathcal{A}$,

$$\phi(\chi_E)(y) = \sum_{\phi(x)=y} \chi_E(x) = \#(\phi^{-1}(y) \cap E)$$

This is the characteristic function of a test $F \in \mathcal{B}$ iff ϕ is an outcome-preserving interpretation, in the sense of Foulis and Randall (1981).

These considerations show that the notions of D-test spaces and generalized test spaces coincide completely.

REFERENCES

Dvurečenskij, A. (1995). Tensor products of difference posets, *Transactions of the American Mathematical Society*, **347**, 1043–1057.

Dvurečenskij, D., and Pulmannová, S. (1994). D-test spaces and difference posets, *Reports on Mathematical Physics*, **34**, 151–170.

Foulis, D. J. (1994). Notes on effect algebras, preprint, University of Massachusetts, Amherst, Massachusetts.

Foulis, D. J., and Bennett, M. K. (1993). Tensor products of orthoalgebras, *Order*, **10**, 271–282.

Foulis, D. J., and Bennett, M. K. (1994). Effect algebras and unsharp quantum logics, *Foundations of Physics*, **24**, 1325–1346.

Foulis, D. J., and Randall, C. H. (1981). Empirical logic and tensor products, in *Interpretations and Foundations of Quantum Theory*, H. Neumann, ed., Wissenschaftsverlag, Mannheim.

Gudder, S. (1988). *Quantum Probability*, Academic Press, San Diego, California.

Hedlíková, J., and Pulmannová, S. (1994). Generalized D-posets and orthoalgebras, preprint.

Kôpka, F., and Chovanec, F. (1994). D-Posets, *Mathematica Slovaca*, **44**, 21–34.

Wilce, A. (1994). Perspectivity and congruence in partial Abelian semigroups, preprint, University of Pittsburgh.